

Moyal Brackets in M-Theory

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Abstract

The infinite limit of Matrix Theory in 4 and 10 dimensions is described in terms of Moyal Brackets. In those dimensions there exists a Bogomol'nyi bound to the Euclideanized version of these equations, which guarantees that solutions of the first order equations also solve the second order Matrix Theory equations. A general construction of such solutions in terms of a representation of the target space co-ordinates as non-local spinor bilinears, which are generalisations of the standard Wigner functions on phase space, is given.

1 Introduction

The purpose of the present paper is to clarify the meaning of the large N limit in M-theory and discuss the features of the formalism peculiar to the situation of 2 and 8 transverse dimensions. When this limit is approached by way of the Moyal Bracket formalism, the matrix M-theory takes on an aspect resembling a generalisation of the Moyal formulation of Quantum Mechanics [14], in terms of Wigner phase space distributions. The target space co-ordinates in 4 and 10-dimensions will be represented in terms of matrix elements between spinor wave functions as Wigner distributions. [1] The non-local character of M-theory [3] is perhaps reflected in this construction. The paper begins with a short review of the usual interpretation of the large N limit before developing the interpretation in terms of Moyal Brackets. The essential simplicity of the theory in 4 and 10 dimensions resides in the fact that the Euclidean form of the Lagrangian can be written as a sum of squares of first order terms with positive relative signs, whose vanishing implies the second order equations of motion, thus giving rise to BPS states. These first order equations may be interpreted as a generalisation of the Liouville equation for distribution functions. This is the basis of the suggested analogy with Quantum Mechanics. This situation is analysed in detail to give a general principle for the construction of solutions to these equations. The 10-dimensional situation is complicated by the presence of equations of constraint. In the 4-dimensional case these constraints are absent, and an alternative solution procedure is outlined.

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2 Setting the Scene

The Lagrangian describing matrix models has been described some time ago [5]. In recent months the paper of Banks et al for a matrix model of M-theory [4] has stimulated the production of a profusion of papers elaborating the issue [6]. We take one such paper, [7] as a convenient reference point. The Lagrangian to be used, taken directly from [7] takes the form of a two-dimensional $\mathcal{N} = 8$ supersymmetric $U(N)$ Yang-Mills theory with the action

$$S = \frac{1}{2\pi\alpha'} \int \text{Tr} \left((D_\mu X)^2 + \theta^T \not{D} \theta + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2} [X^i, X^j]^2 + \frac{1}{g_s} \theta^T \gamma_i [X^i, \theta] \right) d\sigma d\tau. \quad (1)$$

Here the 8 scalar fields X^i are $N \times N$ hermitian matrices, as are the 8 fermionic fields θ_L^α and $\theta_R^{\dot{\alpha}}$. The fields X^i , θ^α , $\theta^{\dot{\alpha}}$ transform respectively as the $\mathbf{8}_v$ vector, and $\mathbf{8}_s$ and $\mathbf{8}_c$ spinor representations of the $SO(8)$ R-symmetry group of transversal rotations. The two dimensional world sheet is a cylinder parametrised by co-ordinates σ , τ , with $0 \leq \sigma \leq 2\pi$. This article goes on to identify vacuum configurations of the theory in the infra-red limit ($g_s \mapsto 0$) with matrices unitarily equivalent to diagonal ones i.e.

$$X^\mu = U^{-1} x^\mu U \quad (2)$$

where x^μ are diagonal matrices with entries x_{ii}^μ . In the simplest description of the model each of the eigenvalues x_{ii} , θ_{ii}^α , $\theta_{ii}^{\dot{\alpha}}$ is considered to represent a Green-Schwarz string in the light cone gauge, and the entire model is described in this limit as a gas of N such strings. In a more general situation envisaged in (1) as σ , the parameter in the compactified direction increases from 0 to 2π , instead of matching $x_{ii}(0)$ with $x_{ii}(2\pi)$ several subsets of the eigenvalues may become permuted and the eigenvalues return to their original values (for a given subset) only after m_k turns round the compact direction. In this case the theory describes $\sum N_k$ strings, where

$$\sum m_k N_k = N. \quad (3)$$

The authors of (1) then proceed to argue that the bulk of the energy in the infinite momentum frame will be carried by long strings in the limit of very large N . There is, however, another way to consider the infinite limit which ends with a theory formulated in terms of Poisson Brackets. This theory possesses some unusual features; the bosonic sector possesses hidden invariances due to Bars [8] which apparently are lost in the finite matrix version, and which represent the remnants of volume preserving diffeomorphisms. Furthermore, when the transverse space is either 2 or 8-dimensional the first order (bosonic) equations of motion can be written which provide a Bogomol'nyi bound on the action. These first order equations furthermore, also possess the additional symmetry. It is to be expected that something similar happens in the case of 4 transverse dimensions, but this case is not analysed in the present article. Another approach to BPS configurations can be found in [9].

3 Passage to the Infinite Limit

The passage to the $N \rightarrow \infty$ limit can be conceived as taking place in two steps; First of all the fields are described by functions of two 'phase space' variables α , β as well as σ , τ which parametrise the world sheet instead of matrices, and expressions of the form $\int \text{Tr}[X^i, X^j]^2 d\sigma$ by

$$\int \left(\frac{1}{\lambda} \sin\{X^i, X^j\} \right)^2 d\alpha d\beta d\sigma \quad (4)$$

where $\sin\{X^i, X^j\}$ is the sine, or Moyal Bracket, [2], with deformation parameter λ which is defined as the imaginary part of the star product (*);

$$X^\mu * X^\nu = \lim_{\substack{\alpha' \rightarrow \alpha \\ \beta' \rightarrow \beta}} e^{i\lambda(\partial_\alpha \partial'_\beta - \partial'_\alpha \partial_\beta)} X^\mu(\alpha, \beta, \sigma) X^\nu(\alpha', \beta', \sigma). \quad (5)$$

The point of this construction is that in the limiting points $\lambda \rightarrow \frac{2\pi}{N}$ the Moyal brackets [10] reproduce the commutators of $N \times N$ matrices through the association of the components X^μ_{mn} of a matrix X^μ with the Fourier modes of a function $X^\mu(\alpha, \beta, \sigma)$, periodic in α, β . (Strictly speaking, N should be odd for this association to work.) The fermionic terms in (1) are replaced by the cosine bracket, i.e. the real part of the star product. Thus the action becomes

$$\begin{aligned} S_{MB} = & \frac{1}{2\pi\alpha'} \int \left((D_\mu X)^2 + \cos\{\theta^T, \not{D}\theta\} + g_s^2 \text{Tr} F_{\mu\nu}^2 \right) d\alpha d\beta d\sigma d\tau \\ & - \int \left(\left(\frac{1}{\lambda g_s^2} \sin\{X^i, X^j\} \right)^2 - \frac{1}{g_s} \cos\{\theta^T \gamma_i, \frac{1}{\lambda} \sin\{X^i, \theta\}\} \right) d\alpha d\beta d\sigma d\tau. \end{aligned} \quad (6)$$

The final step involves letting N grow indefinitely, or equivalently, letting $\lambda \rightarrow 0$. The final form of the action S is then expressed in terms of ordinary Poisson Brackets;

$$\begin{aligned} S_{PB} = & \frac{1}{2\pi\alpha'} \int \left((D_\mu X)^2 + \theta^T, \not{D}\theta + g_s^2 \text{Tr} F_{\mu\nu}^2 - \left(\frac{1}{g_s^2} \{X^i, X^j\} \right)^2 \right) d\alpha d\beta d\sigma d\tau \\ & + \int \left(\frac{1}{g_s} \theta^T \gamma_i, \{X^i, \theta\} \right) d\alpha d\beta d\sigma d\tau. \end{aligned} \quad (7)$$

In this interpretation the theory describes a 3-membrane wrapped around the compactified direction. In order to treat the system dynamically, the longitudinal and timelike modes should be re-instated.

4 Membranes in 8 transverse dimensions

A remarkable feature of the situation with 8 transverse directions is that the theory admits a class of solutions obtainable from a first order formulation, thanks to the existence of a self-dual (antisymmetric) 4-tensor $T_{\mu\nu\rho\sigma}$ in 10-dimensions which is an extension of that discovered in 8-dimensions [11] and which is an analogue of the 4-dimensional fully antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. The equations quoted below are of similar form to those exhibited as equations (63) in [12], but with a more physically appropriate reinterpretation of the nature of the variables. X^0 describes the timelike directions, X^9 the longitudinal and X^i , $i = 1 \dots 8$ the transverse directions. We shall also ignore the dependence upon τ , which is usually interpreted as world-sheet time and treat the additional independent variable as σ . This procedure is tantamount to working in a Euclidean, rather than Lorentzian space, as it is characteristic of equations of self-duality that they should be formulated in a Euclidean space to ensure that solutions will be real. This is the same sign as that adopted in [13].

Then the 10-dimensional membrane has a Lagrangian density in the form of an $SU(\infty)$ pure Yang Mills theory (7). The situation is further simplified to one of dependence upon only one of

the variables σ (apart from the α, β dependence of the gauge potentials $X^\mu(t, \alpha, \beta)$). We work in a gauge where $X^0 = \text{constant}$. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\sigma X^\mu)^2 + \frac{1}{4}\{X^\mu, X^\nu\}^2 \quad (8)$$

where curly brackets denote the Poisson Bracket with respect to the variables α, β , i.e.

$$\{X^\mu, X^\nu\} = \frac{\partial X^\mu}{\partial \alpha} \frac{\partial X^\nu}{\partial \beta} - \frac{\partial X^\mu}{\partial \beta} \frac{\partial X^\nu}{\partial \alpha} \quad (9)$$

and the theory is equivalent to an $SU(\infty)$ Yang Mills with dependence upon only one variable. It is perhaps worth recalling that in the case of finite N Savvidy and others have shown that the classical second order equations for such a theory in the case when that variable is time exhibit chaotic solutions [15]. It is not clear whether this statement persists when the large N limit is taken.

The first order set of equations is then given by

$$\begin{aligned} \partial_\sigma X^1 + \{X^2, X^9\} &= 0 \\ \partial_\sigma X^2 + \{X^9, X^1\} &= 0 \\ \partial_\sigma X^3 + \{X^4, X^9\} &= 0 \\ \partial_\sigma X^4 + \{X^9, X^3\} &= 0 \\ \partial_\sigma X^5 + \{X^6, X^9\} &= 0 \\ \partial_\sigma X^6 + \{X^9, X^5\} &= 0 \\ \partial_\sigma X^7 + \{X^8, X^9\} &= 0 \\ \partial_\sigma X^8 + \{X^9, X^7\} &= 0 \\ \partial_\sigma X^9 + \{X^1, X^2\} + \{X^3, X^4\} + \{X^5, X^6\} + \{X^7, X^8\} &= 0 \\ \{X^1, X^3\} + \{X^4, X^2\} + \{X^5, X^7\} + \{X^8, X^6\} &= 0 \\ \{X^1, X^4\} + \{X^2, X^3\} + \{X^8, X^5\} + \{X^7, X^6\} &= 0 \\ \{X^1, X^5\} + \{X^4, X^8\} + \{X^7, X^3\} + \{X^6, X^2\} &= 0 \\ \{X^1, X^6\} + \{X^2, X^5\} + \{X^3, X^8\} + \{X^4, X^7\} &= 0 \\ \{X^1, X^7\} + \{X^3, X^5\} + \{X^8, X^2\} + \{X^6, X^4\} &= 0 \\ \{X^1, X^8\} + \{X^5, X^4\} + \{X^2, X^7\} + \{X^6, X^3\} &= 0. \end{aligned} \quad (10)$$

It is easy to verify that the second order equations coming from the Lagrangian (8) are satisfied by solutions of this system and that the sums of squares of these equations give the Lagrangian density up to divergences whether the brackets $\{\}$ are taken to signify commutators, Moyal Brackets or Poisson brackets. Thus the solutions provide a bound on the action. In the case of Poisson brackets, in fact only the cross terms involving a σ derivative give divergences; the others vanish identically in virtue of what we call the ‘Saturday Identity’,

$$\{f, g\}\{h, k\} + \{f, h\}\{k, g\} + \{f, k\}\{g, h\} \equiv 0,$$

which holds for Poisson Brackets on a 2-dimensional phase-space-but not for commutators, nor for the Moyal Bracket form. These equations will give rise to solitonic solutions.

What we have in (10) in the case where $X^0 = \text{constant}$ is a set of coupled Nahm equations; When expressed in terms of complex combinations defined by

$$z_1 = X^1 + iX^2; \quad z_2 = X^3 + iX^4; \quad z_3 = X^5 + iX^6; \quad z_4 = X^7 + iX^8; \quad H = X^9, \quad (11)$$

these take the following form

$$\begin{aligned} \partial_\sigma z_1 + i\{H, z_1\} &= 0 \\ \partial_\sigma z_2 + i\{H, z_2\} &= 0 \\ \partial_\sigma z_3 + i\{H, z_3\} &= 0 \\ \partial_\sigma z_4 + i\{H, z_4\} &= 0 \\ \partial_\sigma H + \{\bar{z}_1, z_1\} + \{\bar{z}_2, z_2\} + \{\bar{z}_3, z_3\} + \{\bar{z}_4, z_4\} &= 0 \\ \{z_1, z_2\} + \{\bar{z}_3, \bar{z}_4\} &= 0 \\ \{z_1, z_3\} + \{\bar{z}_4, \bar{z}_2\} &= 0 \\ \{z_1, z_4\} + \{\bar{z}_2, \bar{z}_3\} &= 0. \end{aligned} \quad (12)$$

5 Steps towards a solution

The main purpose of writing the equations in complex form is to emphasise the similarity of the equations to the phase space formulation of Quantum Mechanics.

The observation that the first four equations are of the (Quantum) Liouville type provides a clue towards a possible form of solution. To emphasise the similarity with Quantum Mechanics, the variables with respect to which the brackets are calculated are relabelled as x, p instead of α, β . Take the case where the bracket is Moyal; Then the work of Moyal showed that if $\psi(x, t)$ obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi, \quad (13)$$

then the Wigner distribution function

$$f(x, p, t) = \int_{-\infty}^{\infty} \bar{\psi}(x - y, t) \psi_k(x + y, t) e^{\frac{ipy}{\hbar}} dy \quad (14)$$

satisfies the quantum Liouville equation

$$\frac{\partial}{\partial t} f(x, p, t) = \{H(x, p), f(x, p, t)\}_{MB}, \quad (15)$$

where the bracket is the Moyal bracket. Baker[16] proved a significant converse theorem; that any real valued distribution function satisfying (15) together with a normalisation condition, could be expressed in terms of a Wigner distribution function where the wave function satisfies the Schrödinger equation (13) ². Baker's work suggests that one way to satisfy the equations (12) is to take an ansatz of the form

$$z_k = \int_{-\infty}^{\infty} \chi_k(x - y, \sigma) \psi_k(x + y, \sigma) e^{\frac{2i\pi py}{\hbar}} dy \quad (16)$$

²A more recent discussion of this view of quantum mechanics can be found in[17].

provided ψ_k , $k = 1 \dots 4$ satisfy the Schrödinger equation

$$H(x, \frac{i\partial}{\partial x}, \sigma)\psi_k(x, \sigma) = \frac{i\lambda}{2\pi} \frac{\partial}{\partial \sigma} \psi_k(x, \sigma) \quad (17)$$

and $\chi_k(x, \sigma)$ satisfy the conjugate equation. This result follows from the observation that

$$\begin{aligned} & \frac{i}{2}(H * z_k - z_k * H) \\ &= \sum_j \sum_k (-\lambda)^{m+n} \frac{\partial_x^m}{m!} \frac{\partial_p^n}{n!} H(x, p, \sigma) (-\partial_x)^n \int_{-\infty}^{\infty} y^m \chi_k(x-y, \sigma) \psi_k(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy \\ & \quad - H \leftrightarrow z_k \\ &= \int_{-\infty}^{\infty} H(x+y, p+i\partial_x) \chi_k(x-y, \sigma) \psi_k(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy - H \leftrightarrow z_k \\ &= \int_{-\infty}^{\infty} H(x+y, i\partial_{x+y}) \chi_k(x-y, \sigma) \psi_k(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy \\ & \quad - \int_{-\infty}^{\infty} H(x-y, i\partial_{x-y}) \chi_k(x-y, \sigma) \psi_k(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy. \end{aligned} \quad (18)$$

In a similar fashion the products $z_j * z_k$ can be calculated thanks to the following Lemma³.

$$\begin{aligned} e^{py} f(x) * e^{py'} g(x) &= \sum_j \sum_k e^{p(y+y')} (-\lambda)^{m+n} \frac{(y' \partial_x)^m}{m!} f(x) \frac{(-y \partial_x)^n}{n!} g(x) \\ &= e^{p(y+y')} f(x+y') g(x-y) \end{aligned} \quad (19)$$

Then, by definition

$$z_j * z_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_j(x-y, \sigma) \psi_j(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} \chi_k(x-y', \sigma) \psi_k(x+y', \sigma) e^{\frac{2i\pi p y'}{\lambda}} dy dy' \quad (20)$$

This gives after integration over the variable $y - y'$

$$z_j * z_k = N_{jk} \int_{-\infty}^{\infty} \chi_j(x-y-y', \sigma) \psi_k(x+y+y', \sigma) e^{\frac{2i\pi p(y+y')}{\lambda}} d(y+y'), \quad (21)$$

where N_{jk} is a (σ dependent) normalisation factor;

$$N_{jk} = \int_{-\infty}^{\infty} \chi_k(x-u, \sigma) \psi_j(x+u, \sigma) du. \quad (22)$$

There remain the $6 = 3 +$ conjugate equations of constraint. One obvious way in which these can be satisfied is to choose $N_{jk} = n_j \delta_{jk}$. This then leaves one further equation to be solved, namely,

$$\partial_\sigma H + \{\bar{z}_1, z_1\} + \{\bar{z}_2, z_2\} + \{\bar{z}_3, z_3\} + \{\bar{z}_4, z_4\} = 0. \quad (23)$$

This equation determines the σ dependence of H . However this procedure gives only a limited class of solutions, except in the 4-dimensional situation, which will be analysed again later. In order to solve the full 10-dimensional constraints in general, it is necessary to modify the ansatz to take into account the group theoretical structure of the self-duality relations in 8-dimensions.

³This was suggested by an unpublished result of Ian Strachan

6 Solution of 10-d equations

Suppose γ^j , $j = 1 \dots 8$ are the gamma matrices in 8-dimensions, which admit a real representation by 16×16 antisymmetric matrices [11] and ψ is a 16 component spinor, which will be later subject to algebraic constraints. Take as ansatz

$$\begin{aligned} X^k &= i \int_{-\infty}^{\infty} \bar{\psi}(x-y, \sigma) \gamma^k \psi(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy, \quad k = 1 \dots 8 \\ X^9 &= i \int_{-\infty}^{\infty} \bar{\psi}(x-y, \sigma) \tilde{\gamma}^9 \psi(x+y, \sigma) e^{\frac{2i\pi p y}{\lambda}} dy, \quad \alpha = 9 \end{aligned} \quad (24)$$

and leave x^9 as an unspecified anti-Hermitean matrix for the moment. By construction all X^k , $k = 1 \dots 8$, are real. There is nothing in this choice of which violates any symmetry principle; we are already dealing with a broken symmetry situation. Consider the six equations of constraint in (10), which are identical to six of the seven equations of self-duality in 8-dimensions proposed in [11]. It is relatively easy to see that these last six will be satisfied automatically if ψ is constrained to be of the form

$$\psi = P_1 P_2 P_3 \tilde{\psi} \quad (25)$$

where $\tilde{\psi}$ is arbitrary and P_1 , P_2 , P_3 are three mutually commuting projection operators given by

$$\begin{aligned} P_1 &= \frac{1}{4}(1 + \gamma^1 \gamma^3 \gamma^5 \gamma^7)(1 + \gamma^2 \gamma^4 \gamma^6 \gamma^8) \\ P_2 &= \frac{1}{4}(1 + \gamma^1 \gamma^4 \gamma^8 \gamma^5)(1 + \gamma^2 \gamma^3 \gamma^7 \gamma^6) \\ P_3 &= \frac{1}{4}(1 + \gamma^1 \gamma^6 \gamma^4 \gamma^7)(1 + \gamma^2 \gamma^5 \gamma^3 \gamma^8). \end{aligned} \quad (26)$$

This leaves the equations involving σ dependence; the most conspicuous among which is

$$\partial_\sigma X^9 = -(\{X^1, X^2\} + \{X^3, X^4\} + \{X^5, X^6\} + \{X^7, X^8\}). \quad (27)$$

The other equations involving σ derivatives and (27) are then solved with the ansatz;

$$\tilde{\gamma}^9 = -(\gamma^1 \gamma^2 + \gamma^3 \gamma^4 + \gamma^5 \gamma^6 + \gamma^7 \gamma^8), \quad (28)$$

if ψ has an exponential σ dependence. It might be thought that this method would lead to a general construction of a σ independent solution, by choosing instead of (27)

$$P_4 = \frac{1}{4}(1 + \gamma^1 \gamma^6 \gamma^2 \gamma^5)(1 + \gamma^4 \gamma^7 \gamma^3 \gamma^8), \quad (29)$$

and choosing as projection

$$\psi = P_1 P_2 P_4 \tilde{\psi} \quad (30)$$

Then not only do the equations of constraint vanish, but so does the right hand side of (27) and we should have, in principle, a solution to the 8-dimensional self dual equations. However, this is not so, as it may be verified that the product $P_1 P_2 P_4 \equiv 0$, i.e. the projection operator vanishes! By taking only two projection operators, P_2 and P_3 in the construction it is found that only 5 of the 6 equations of constraint vanish. The consistency of the system may then be restored by introducing another co-ordinate X^0 , which plays a similar role to X^9 .

7 Digression on the 4-d Moyal Nahm equations

There is an alternative way to look at the 4-dimensional version of the first order equations, which is closer in spirit to the initial discussion in which the equations are regarded as an extension of Quantum Mechanics. This relationship is obscured in the 10-dimensional case on account of the equations of constraint. Consider the 4-dimensional Moyal-Nahm equations, where again X^0 is set to a constant

$$\begin{aligned}\partial_\sigma X^1 + \{X^2, X^3\} &= 0 \\ \partial_\sigma X^2 + \{X^3, X^1\} &= 0 \\ \partial_\sigma X^3 + \{X^1, X^2\} &= 0.\end{aligned}\tag{31}$$

and assume that the functions $\psi_j(x)$ belong to a real orthonormal basis over the real line. Then take as ansatz

$$X^i = \sqrt{(E_j - E_i)(E_i - E_k)} e^{(E_j - E_k)\sigma} \int_{-\infty}^{\infty} \psi_j(x-y) e^{\frac{2i\pi py}{\lambda}} \psi_k(x+y) dy \tag{32}$$

together with cyclic replacements. The E_j are constants. Then on account of the result (21) the Moyal-Nahm equations are satisfied identically. This solution can be extended in an obvious manner by separating the functions into three classes according to the residue of the index mod 3;

$$\begin{aligned}X^0 &= \sum_{j=0}^{j=\infty} \sqrt{(E_{3j+1} - E_{3j})(E_{3j} - E_{3j+2})} e^{(E_{3j+1} - E_{3j+2})\sigma} \int_{-\infty}^{\infty} \psi_{3j+1}(x-y) e^{\frac{2i\pi py}{\lambda}} \psi_{3j+2}(x+y) dy \\ X^1 &= \sum_{j=0}^{j=\infty} \sqrt{(E_{3j+2} - E_{3j+1})(E_{3j+1} - E_{3j})} e^{(E_{3j+2} - E_{3j})\sigma} \int_{-\infty}^{\infty} \psi_{3j+2}(x-y) e^{\frac{2i\pi py}{\lambda}} \psi_{3j}(x+y) dy \\ X^2 &= \sum_{j=0}^{j=\infty} \sqrt{(E_{3j} - E_{3j+2})(E_{3j+2} - E_{3j+1})} e^{(E_{3j} - E_{3j+1})\sigma} \int_{-\infty}^{\infty} \psi_{3j}(x-y) e^{\frac{2i\pi py}{\lambda}} \psi_{3j+1}(x+y) dy\end{aligned}$$

The link with the above discussion on the analogy with the phase space formulation of Quantum Mechanics is as follows; When the correspondence $p \rightarrow \frac{i\lambda\partial_x}{2\pi}$ is implemented

$$\begin{aligned}X^3 \psi_1(x) e^{-E_1\sigma} &= \sqrt{(E_1 - E_3)(E_3 - E_2)} e^{-E_2\sigma} \int_{-\infty}^{\infty} \psi_1(x-y) e^{\vec{y}\vec{\partial}_x} \psi_2(x+y) dy \psi_1(x) \\ &= \sqrt{(E_1 - E_3)(E_3 - E_2)} e^{-E_2\sigma} \int_{-\infty}^{\infty} \psi_1(x-y) \psi_2(x) \psi_1(x-y) dy \\ &= \sqrt{(E_1 - E_3)(E_3 - E_2)} \psi_2(x) e^{-E_2\sigma}\end{aligned}\tag{33}$$

on account of the orthogonality relations. Similarly, exploiting the substitution $y \mapsto -y$ in the integral in (32) a similar equation for the adjoint appears;

$$\begin{aligned}\psi_2(x) e^{E_2\sigma} \bar{X}_3 &= -\sqrt{(E_1 - E_3)(E_3 - E_2)} e^{E_1\sigma} \int_{-\infty}^{\infty} \psi_2(x) \psi_1(x+y) e^{-\overleftarrow{y}\overleftarrow{\partial}_x} \psi_2(x-y) dy \\ &= -\sqrt{(E_1 - E_3)(E_3 - E_2)} \psi_1(x) e^{E_1\sigma} \int_{-\infty}^{\infty} \psi_2(x-y) \psi_2(x-y) dy \\ &= -\sqrt{(E_1 - E_3)(E_3 - E_2)} \psi_1(x) e^{E_1\sigma}\end{aligned}\tag{34}$$

This pair of equations can be combined into a Schrödinger like form with X_3 playing the role of Hamiltonian.

7.1 Back to Matrix Models

The treatment of the Moyal version of these first order Matrix Theory equations given here allows a neat return to the Matrix formulation in the following manner; define

$$X^k = \sum_m \sum_n A_{mn}^k(t) \int_{-\infty}^{\infty} \psi_m(x-y) \psi_n(x+y) e^{\frac{2i\pi py}{\lambda}} dy \quad (35)$$

where the coefficients A_{mn}^k are σ or τ dependent elements of infinite-dimensional matrices labelled by the same index as X^k . The use of the Lemma (21) transforms the Moyal version of equations (10) into the infinite matrix version of the equations. This manipulation shows how closely the infinite matrix formulation is tied to the Moyal treatment.

8 Conclusions

This article has been written to examine the Moyal version of Matrix theory, in the belief that this formulation is the most appropriate for the discussion of the large N limit and for the investigation of parallels with Quantum Mechanics. The essence of the solution to the first order equations lies in the representation of the co-ordinates in terms of matrix elements of a spinor in 8-dimensions. The construction carries echoes of a similar representation of the 10-dimensional string in terms of bilinears [18], but here a noteworthy feature of the construction is that it is intrinsically non-local. It would appear that instead of ‘magic, mystery, or membrane’ which have been suggested as a possible etymology [19], the M in M-Theory really stands for Moyal!

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